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Quantum Logic and Decoherence

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The quantum logical approach to quantum mechanics in Hilbert space presupposes value definiteness of elementary propositions. Although the description of the measurement process by sequential quantum logic seems to justify this precondition, it is found to be incompatible with the quantum theory of measurement, which does not provide the decoherence of pointer values. The attempts to solve the measurement problem by means of histories and by quantum gravity fail, since these approaches are based on sequential quantum logic and its preconditions, too. Finally, we discuss consequences of these results.

KEY WORDS: quantum logic; decoherence; objectification.

1. INTRODUCTION

For the Hilbert space H_S of a proper quantum system S the algebra of projection operators P_A, P_B, ... which project onto closed linear manifolds M_A, M_B, ... is given by the Hilbert lattice L_H. Except from more complicated properties (Solér, 1995), L_H is a complete, orthomodular, and irreducible lattice L_Q with elements 0 and 1. In addition L_H is atomic and fulfills the *covering law* (Jauch, 1968). If these properties are included we denote the lattice by L_Q^*.

It is a most remarkable result that the lattice L_Q can be reconstructed as the lattice of propositions which are elementary, value definite, and restrictedly available. If the propositions attribute properties to an individual system, then we can reconstruct even the lattice L_Q^*. This approach to quantum mechanics, called *quantum logic*, can be extended by the additional *Solér* property (Solér, 1995) leading to the Hilbert lattice L_H and finally to the classical Hilbert spaces, making use of a theorem by *Piron* (Piron, 1976).

The quantum logical reconstruction of quantum mechanics in Hilbert space suggests that within the framework of Hilbert space quantum mechanics we can justify those conditions which were presupposed for elementary propositions. These conditions are *objectification* in the measurement process and *decoherence* of the

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macroscopic pointer properties. However, there are serious difficulties to fulfill these conditions in Hilbert space quantum mechanics—which are well known as the measurement problem.

In order to preserve the compatibility of quantum logic with the pragmatic preconditions of the formal language we will consider here several attempts to avoid the inconsistency between the quantum logical approach to quantum mechanics and the lack of decoherence induced by quantum mechanics. In particular, we mention here the quantum logical formulation of the history approach by *Isham* (Isham, 1994), and the attempt to solve the measurement problem by means of gravity (Penrose, 1999(a)). Since both approaches do not lead to convincing solutions we will finally discuss some new and not yet elaborated proposals.

2. QUANTUM LANGUAGE AND QUANTUM LOGIC

2.1. Language, Semantics, and Pragmatics

Let S be a proper quantum system and A, B, ... elementary propositions which attribute predicates $P(A)$, $P(B)$, ... to system S at times $t_1, t_2, \ldots$. Hence, we write for the elementary propositions $A(S, t_1)$, $B(S, t_2)$, ... We will assume here, that for every elementary proposition A there exists a finite testing procedure which shows whether or not $P(A)$ pertains to S. If $P(A)$ pertains to S at time t_1, then the proposition $A(S, t_1)$ is called to be "true," otherwise $A(S, t_1)$ is said to be "false." The assumption, that for every elementary proposition there is a testing procedure which decides between "true" and "false" means, that these propositions are "value definite." Hence, an elementary proposition can either be proved (with result A) or disproved (with result $\bar{A}$), where $\bar{A}$ is the counter proposition of A.

Furthermore, we assume that after a successful proof of A a new proof attempt leads with certainty to the same result, provided the time interval between the two proof attempts is sufficiently small. This assumption means that there are repeatable measurement processes, which can be applied to the testing procedures. However, if after a successful proof of A, say, another proposition B is proved, then a new proof attempt will in general not lead to the previous positive result. Hence, we will not assume that two propositions A and B are in general simultaneously decidable. If accidentally two propositions A and B are always jointly decidable, we will call A and B to be "commensurable." In this case, after the proof attempt of B the result of the previous A–test is still available. However, in the general case the result of a previous test is only restrictedly available.

On the basis of the set $\mathfrak{S}_Q^e$ of elementary propositions we introduce the logical connectives by the possibilities to attack or to defend them, i.e. by the possibilities to prove or to disprove the connective. Here, we consider the sequential conjunction $A \sqcap B$ (A and then B) which refers to two subsequent instants of time t_1 and t_2 with $t_1 < t_2$ and the logical connectives $\neg A$ (not A), $A \wedge B$ (A and B), $A \vee B$

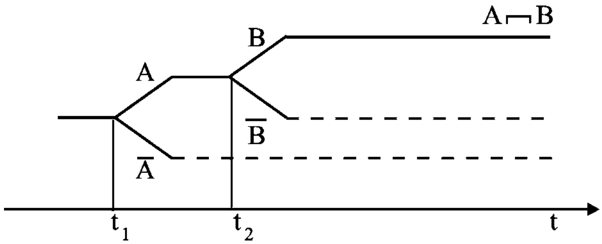

Fig. 1. Proof–tree for the sequential conjunction $A \sqcap B$. $\bar{A}$ and $\bar{B}$ are the counter propositions to A and B.

(A or B), and $A \to B$ (if A then B)—which refer to one simultaneous instant of time. The definitions of the sequential and logical connectives by attack- and defence schemes can be illustrated most conveniently by chronologically ordered proof trees. Correspondingly, in the proof tree of the sequential conjunction $A \sqcap B$ as shown in Fig. 1, the first branching point corresponds to a A–test at t_1, the second one to a B–test at t_2.

Note, that for the truth of $A \sqcap B$ the commensurability of A and B does not matter. However, for the proof trees of the logical connectives, which refer to one simultaneous instant of time, the commensurabilities of the elementary propositions play an important role. The concepts of truth and falsity of a compound proposition which is composed by the connectives can then be defined by success and failure in a proof tree, respectively.—For the details of this well established operational approach we refer to the literature (Mittelstaedt, 1978, 1987; Stachow, 1980).

Furthermore, we will define here binary relations between propositions. First, the proof equivalence $A \equiv B$ means that A can be replaced in any proof tree of a compound proposition by B without thereby changing the result of the proof tree. Second, the value equivalence $A = B$ means that A is true (in the sense of a proof tree) if and only if B is true. Third, the relation of implication $A \leq B$ can be defined by $A \equiv A \wedge B$. Hence, the two implications $A \leq B$ and $B \leq A$ imply the proof equivalence $A \equiv B$. Finally, we mention that $A \to B$ is true if and only if $A \leq B$ holds.

The full quantum language $\mathfrak{S}_Q$ can then inductively be defined by the set $\mathfrak{S}_Q^e$ of elementary propositions and the connectives mentioned. Together with the always true elementary proposition V, the always false elementary proposition Λ, and the three relations the language $\mathfrak{S}_{\mathrm{Q}}$ reads

$$\mathfrak{S}_Q = \left\{\mathfrak{S}_Q^e; \sqcap, \wedge, \vee, \to, \neg; V, \Lambda; \equiv, =, \leq \right\}. \tag{1}$$

The connectives are defined by attack- and defence schemes which can be illustrated by proof trees. In addition, it is important to note that for the sequential and the logical connectives there are value equivalent elementary propositions, the measurement devices of which can explicitly be constructed (Stachow, in press). Hence, by induction with respect to the connectives one arrives at the result that for any finitely connected proposition A there is an elementary proposition A^e which is value equivalent to A, i.e. which satisfies $A = A^e$. For the algebraic structure of the language $\mathfrak{S}_Q$ these value equivalent elementary propositions play an important role.

2.2. Quantum Logic

The semantics described here is a combination of a realistic semantics (for elementary propositions) and a proof semantics (for connectives). Hence, the truth of a compound proposition depends on the connectives contained in it as well as on the elementary propositions and their truth values. However, there are finitely connected propositions which are true in the sense of the semantics mentioned, irrespective of the truth values of the elementary propositions contained in it. These propositions are called *formally true*.—The precondition that measurements are repeatable implies that $A \to A$, the *law of identity*, is formally true. The value definiteness of elementary propositions implies that also finitely connected propositions are value definite and thus $A \vee \neg A$, the *tertium non datur law*, is formally true. In a similar way, it follows that $\neg(A \wedge \neg A)$, the *law of contradiction*, and $(A \wedge (A \to B)) \to B$, the *modus ponens law*, are formally true.—Formally true propositions can also be expressed by "formally true implications." E.g. the modus ponens law reads $A \wedge (A \to B) \leq B$. In addition, if we make use of the special propositions V (verum) and Λ (falsum), then the relations $A \leq V$ and $\Lambda \leq \mathrm{A}$ hold for all propositions $A \in \mathfrak{S}_Q$. The formal truth of a proposition A can then be expressed by $V \leq A$. E.g. the tertium non datur law reads $V \leq A \vee \neg A$ and the law of contradiction $A \wedge \neg A \leq \Lambda$.

There are two kinds of propositions $A \in \mathfrak{S}_Q$. If a compound proposition contains in addition to elementary and commensurability propositions only the *logical* connectives $\wedge$, $\vee$, $\wedge$, and $\to$, then it is called a "logical proposition." In the more general case, when the proposition contains also *sequential* connectives, in particular the sequential conjunction $\sqcap$, then it is called a "sequential proposition." In addition to the formally true logical propositions mentioned above, there are also formally true sequential propositions. If A and B are logical propositions then $A \wedge B \leq A \sqcap B$ is a formally true implication. The totality of formally true implications can be summarised in a calculus which contains "beginnings" $\Rightarrow A \leq B$ and rules $A \leq B \Rightarrow C \leq D$. Here, we distinguish the calculus $\mathbf{L}_Q$ of formally true *logical* propositions and the calculus $\mathbf{S}_Q$ of formally true sequential propositions. For the explicit form of these formal systems we refer to the literature (Mittelstaedt, 1978; Stachow, 1980).

For an algebraic characterisation of the calculi $\mathbf{L}_Q$ and $\mathbf{S}_Q$ we consider the corresponding Lindenbaum–Tarski algebras, i.e. the algebra of equivalence classes which is given here by the algebra of value equivalent elementary propositions. The Lindenbaum–Tarski algebra of the calculus $\mathbf{L}_Q$ is given by a complete, orthomodular lattice L_Q. Subsets of mutually commensurable propositions constitute a Boolean sublattice $L_B \subseteq L_Q$ of the lattice L_Q (Mittelstaedt, 1987). Moreover, if the entire quantum language $\mathfrak{S}_Q$ refers to one individual quantum system, then the lattice L_Q is atomic (where the atoms correspond to pure states) and fulfills the covering law (Stachow, 1984). In this case we denote the lattice by L_Q^*. The Hilbert lattice L_H of projection operators in Hilbert space (Birkhoff and von Neumann, 1936) can be obtained from the lattice L_Q^* by adding the Solér law, the meaning of which is, however, still open (Solér, 1995). Correspondingly, the Lindenbaum–Tarski algebra of the calculus $\mathbf{S}_Q$ of sequential quantum logic is given by a Baer* semigroup. We will not go into details here and refer to the literature (Foulis, 1960; Stachow, 1980). It is well known that by means of a result by *Piron* (Piron, 1976) from the lattice L_H the three classical Hilbert spaces can be obtained and that for the complex numbers $\mathbf{C}$ quantum mechanics in Hilbert space is achieved.

3. QUANTUM THEORY OF MEASUREMENT

3.1. Quantum Logical Description of the Measurement Process

The quantum theory of measurement in the version of J. von Neumann which presupposes the objectification of the measurement results as the *projection postulate* can be reformulated in terms of quantum logic. Let $S = S(W)$ be a proper quantum system S with Hilbert space H_S and with the preparation $W \in L_Q$, where the atomic proposition W describes a pure state corresponding to a projection operator $P[\varphi]$, $\varphi \in H_\mathrm{S}$. For the experimental test of another elementary proposition A measurement process must be performed which has two possible outcomes, A and $\neg A$. This means that after the first step of a measurement process one of the two sequential propositions $W \sqcap A$ or $W \sqcap \neg A$ is true for system S. If the conditional probabilities for A and $\neg A$ are given by $p(W, A)$ and $p(W, \neg A)$, respectively, after this first step of the measurement process the system is described by the ensemble $\Gamma(W; A) = \{p(W, A), p(W, \neg A); W \sqcap A, W \sqcap \neg A\}$ of two weighted alternatives. The ensemble $\Gamma(W; A)$ represents the observers knowledge about S after step (I) of the measurement process.

In a second step (II) this ensemble can be reduced merely by reading to one of the possible outcomes $W \sqcap A$ or $W \sqcap \neg A$, respectively. Hence, the whole measurement process with result A, say, reads $W \Rightarrow_{(I)} \Gamma(W; A) \Rightarrow_{(\mathrm{II})} W \sqcap A$. In step (I) the system $S(W)$ is changed such that proposition A is objectified, whereas in step (II) the observers ignorance about S is removed. Hence, in the proof tree of the elementary proposition A, shown in Fig. 2, step (I) corresponds to the transition from the preparation W to the classical mixture $\Gamma(W; A)$ of sequential propositions,

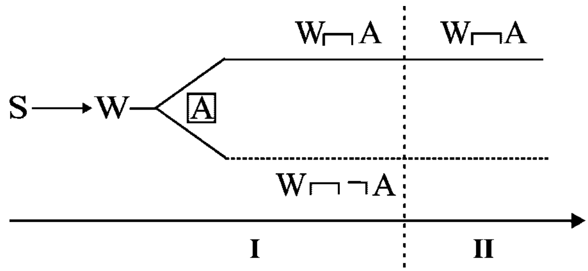

Fig. 2. Proof-tree of proposition A in quantum logic.

whereas the step (II) describes the reduction of this mixture of propositions to $W \sqcap A$ or to $W \sqcap \neg A$ by reading.

3.2. Decoherence and the Measurement Problem

The quantum theory of measurement does not reproduce the description of the measurement process given in the previous subsection. In order to explain this surprising statement, let us briefly recall the main steps of the contemporary quantum theory of measurement (Busch *et al.*, 1996; Mittelstaedt, 1998). Accordingly, the measurement process consists of three steps, *1. preparation, 2. premeasurement,* and *3. objectification and reading*. If the object system is prepared in a pure state $\varphi(S)$ and the apparatus M in a neutral state $\Phi(M)$, then the compound system $S + M$ is prepared in the state $\Psi(S + M) = \varphi(S) \otimes \Phi(M)$. Here we consider a repeatable unitary premeasurement of a discrete observable $A = \sum a_i P[\varphi_i]$ with eigenstates φ_i and eigenvalues a_i. In a second step a unitary operator U_A is applied to the state $\Psi(S + M)$ such that

$$U_A \Psi(S + M) = \Psi'(S + M) = \sum c_i \varphi_i \otimes \Phi_i \tag{2}$$

with $c_i = (\varphi_i, \varphi)$ and eigenstates Φ_i of the pointer observable $Z = \sum Z_i \cdot P[\Phi_i]$. In order to achieve this special biorthogonal decomposition of the state Ψ' the unitary operator U_A must fulfill the following calibration condition: If the system S is prepared in an eigenstate φ_i of A, then the post-premeasurement state reads $\Psi' = \varphi_i \otimes \Phi_i$.

If after the premeasurement when the compound system is in the state $\Psi'(S + M)$ the system S and the apparatus M are considered as separate objects, then these systems have to be described by the correlated mixed states

$$W'_S = \sum |c_i|^2 P[\varphi_i] \quad \text{and} \quad W'_M = \sum |c_i|^2 P[\Phi_i], \tag{3}$$

respectively. However, it is a most important theoretical result that these mixed states do not admit ignorance interpretation. This means that the states W'_S and W'_M must not be considered as weighted mixtures of states $\Gamma'_S := \{\varphi_i, |c_i|^2\}$ and

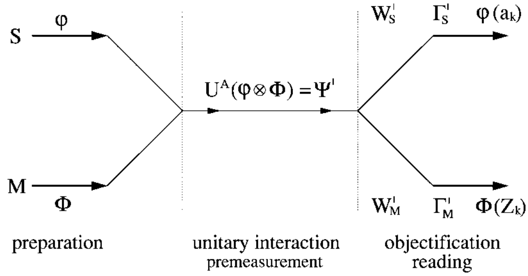

Fig. 3. Schematic representation of the measurement process in quantum mechanics.

$\Gamma'_M := \{\Phi_i, |c_i|^2\}$, such that S is actually in state φ_n, say, and the observer doesn't know this state but only its probability $p(\varphi, a_n) = |c_n|^2$. Analogously, the apparatus M is not actually in a state Φ_n and the observer knows only its probability. Hence, after the premeasurement the values a_i and Z_i of the object observable A and of the pointer observable Z are objectively undetermined and not merely subjectively unknown. (Fig. 3)

However, the final step of the measurement process, reading the results a_i and Z_i, presupposes that these values are objectively determined. Hence, there must be some real process that transforms the mixed states W'_S and W'_M into the weighted mixtures of states Γ'_S and Γ'_M, respectively. This process, *the objectification*, is not contained in the quantum theory of measurement mentioned. Instead, it is added to this theory as a new postulate or as a hypothetical assumption. J. von Neumann called it "projection postulate." With respect to the objectification of the values Z_i of the *macroscopic* pointer, the transition from W'_M to Γ'_M is called "decoherence."

These considerations show that the quantum logical description of the measurement process is not reproduced by the quantum theory of unitary premeasurements. Indeed, it is not clear how the transition from the preparation $W = P[\varphi]$ to the mixture $\Gamma(W, A)$ of sequential propositions really works. Hence, step (I) of the proof tree, the "objectification," does not correspond to a realisable quantum mechanical process. In particular, this means that elementary propositions cannot be tested in the described way and are thus not *value definite*. Hence, the nonobjectifiability of pointer values, or the lack of decoherence precludes the basic assumptions of value definitness and finite decidability of quantum logical propositions.

4. ATTEMPTS TO ACHIEVE POINTER OBJECTIFICATION

4.1. The History Approach

There is an interesting attempt to overcome the problem of objectification in the measurement process which goes back to Wigner (Wigner, 1971). Since quantum mechanics does not lead to an adequate description of the measurement outcomes, von Neumann introduced the "projection postulate" as an additional requirement. Wigner tried to present a formulation of quantum mechanics without the *ad hoc* introduced projection postulate. For this reason he admits that quantum mechanics gives only probability connections between successive observations on quantum systems. Obviously, this formulation gives some primacy to the act of observation which is considered as an irreducible basic concept in quantum mechanics.

Let $W = W(t_0)$ with $W = W^+, \mathrm{tr}\{W\} = 1$ be the preparation state of a system S at time $t = t_0$, and P_A, P_B, and P_C observables (projection operators) with eigenvalues $(A, \neg A)$, $(B, \neg B)$, and $(C, \neg C)$ which are measured successively at times $t_1 < t_2 < t_3$, $(t_1 > t_0)$, respectively. If the successive measurements are not carried out immediately but at times t_1, t_2, t_3 with finite intervals the projection operators have to be transformed by some unitary development operator

$$U(t, t') = e^{-i/\hbar H(t-t')}, \quad \text{e.g.} \tag{4}$$

$$P_B(t_2) = e^{-i/\hbar H(t_2-t_1)} P_B(t_1) e^{i/\hbar H(t_2-t_1)}. \tag{5}$$

The probability to find the eigenvalue A after the first measurement is given by

$$p(W; P_A(t_1)) = \mathrm{tr}\{W(t_0) P_A(t_1)\} \tag{6}$$

and the post-measurement (Lüders) state reads

$$W'(t_1) = \frac{P_A(t_1) W(t_0) P_A(t_1)}{\mathrm{tr}\{W(t_0) P_A(t_1)\}}. \tag{7}$$

The conditional probability for $B(t_2)$ given that $A(t_1)$ was found then reads

$$p(W'; P_B(t_2)) = \frac{\mathrm{tr}\{P_B(t_2) P_A(t_1) W(t_0) P_A(t_1) P_B(t_2)\}}{\mathrm{tr}\{W(t_0) P_A(t_1)\}}. \tag{8}$$

Finally, we find the probability to get first $A(t_1)$ and then $B(t_2)$ by multiplying this expression with $p(W; P_A(t_1))$

$$p(W(t_0); A(t_1) \sqcap B(t_2)) = \mathrm{tr}\{P_B(t_2) P_A(t_1) W(t_0) P_A(t_1) P_B(t_2)\} \tag{9}$$

where we have used the notation $A \sqcap B$ known from sequential quantum logic. The generalization of this formula to three and more measurements leads to

$$\begin{aligned} &p(W(t_0); A(t_1) \sqcap B(t_2) \sqcap C(t_3)) = \\ &\mathrm{tr}\{P_C(t_3) P_B(t_2) P_A(t_1) W(t_0) P_A(t_1) P_B(t_2) P_C(t_3)\} \end{aligned} \tag{10}$$

etc. and is known as "Wigner's formula." Wigner hoped to avoid the problems connected with the collapse of the state vector, if quantum mechanics is restricted to probability statements of this kind. We will comment this conjecture at the end of this section.

It is obvious, that the probability of "Wigner's formula" refers to the preparation $W(t_0)$ and to the sequential proposition $Q = A(t_1) \sqcap B(t_2) \sqcap C(t_3)$ which is given by the product $P_C(t_3) \cdot P_B(t_2) \cdot P_A(t_1)$ of projection operators in Hilbert space. Hence, the subject of Wigner's probabilities are sequences of time ordered projection operators, or of time ordered propositions. These expressions are also called "*histories*," or more precisely "*homogeneous histories*." The intimate relation between sequential propositions and "histories" was first elaborated by Isham (Isham, 1994) in 1994. It must be emphasised that most studies in quantum logic consider propositions at a single time and are therefore not applicable to the present problem. However, quantum logic is applicable to the history approach, if in addition to the theory mentioned the logic of sequential propositions is used. Indeed, as pointed out by Isham, a "history filter" is a time-labelled version of a "sequential conjunction."

Histories were used by many authors as a means for describing the phenomenon of decoherence in large quantum systems without thereby being in conflict with quantum mechanics. We mention here in particular the work of Gell-Mann and Hartle (Gell-Mann and Hartle 1993; Hartle, 1995), Omnes (Omnes, 1994), and Kiefer (Kiefer, 1996). However, histories provide merely a description of the decoherence phenomenon but not its justification. Indeed, if histories are written explicitly as a time-ordered sequence

$$h(A) = A_1(t_1) \sqcap A_2(t_2) \sqcap \ldots; t_1 \leq t_2 \leq \ldots \tag{11}$$

of elementary propositions, then it becomes clear that the objectification in the measurement is already presupposed. The sequential conjunction $A(t_1) \sqcap B(t_2)$ is defined by two successive measurements at times t_1 and t_2, respectively. For every measurement of this kind the objectification of the final results must be presupposed. Hence, histories cannot be used for justifying the objectification and decoherence.

4.2. Quantum Gravity

A different way of reasoning refers to quantum gravity. There is a new interest in sequential quantum logic and its extension to the history approach since some authors consider it as a basis for constructing quantum gravity (Gell-Mann and Hartle, 1993; Hartle, 1995; Isham, 1994). For this program quantum logic must first be generalised to relativistic quantum logic (Mittelstaedt, 1983). We will not go here into details. Of course, quantum gravity has its own interest. For the present problem of objectification and decoherence it becomes relevant, since quantum gravity could perhaps help understanding the measurement problem. Penrose (Penrose, 1999a,b) and others argue in favour of *gravity induced superselection rules*. An experiment, recently proposed by Penrose (Penrose, 1995b) could provide perhaps a clear-cut test of whether or not objectification is a gravitational phenomenon.

However, sequential quantum logic as well as the history approach presuppose objectification, since sequential conjunctions are time–ordered sequences of *objective* facts. Since quantum mechanics leads to the nonobjectifiability of the measurement outcomes, this result obviously invalidates also sequential quantum logic, the history approach, and quantum gravity based on it. Hence, quantum gravity cannot be used for explaining the objectification by gravity induced superselection rules, simply since objectification was already presupposed. Obviously, this way of reasoning is a vicious circle.

5. CONCLUDING REMARKS

Is quantum logic fundamental? We have seen that the most general pragmatic preconditions of a scientific language imply quantum logic and that quantum logic leads—together with some additional formal conditions—to the Hilbert space and to quantum mechanics. Quantum mechanics is *consistent* with the pragmatic preconditions of quantum logic in the sense that there is no general commensurability of quantum mechanical propositions. However, quantum mechanics is *inconsistent* with the pragmatics in the sense that it does not lead to the objectification of system values after premeasurement. Hence, quantum mechanics cannot justify the value definiteness of elementary propositions which was presupposed in the pragmatics.

The quantum theory of unitary premeasurements cannot explain the decoherence of the pointer values. The attempt to explain decoherence by means of consistent histories fails since histories correspond to sequential propositions which already presuppose the objectification in the measurement process. In addition, if histories are used as a means to formulate quantum gravity, the influence of gravity on the measurement process cannot be used for the derivation of some gravity induced pointer objectification. Hence, within the framework of the present considerations we cannot find out where the decoherence in the measurement process comes from.

If the objectification in the measurement process cannot be explained by the quantum theory of unitary premeasurements then it suggests itself to begin with elementary propositions that are not value definite. A formal language and logic of not necessarily value definite quantum mechanical propositions can be constructed in various ways, either in analogy to intuitionistic logic (Mittelstaedt, 1978) or by a modification of the algebraic structure (Dalla Chiara, 1995; Giuntini, 1990). However, it is still an open question whether in this way a consistent operational approach to quantum mechanics can be obtained. First, the weak algebraic structures (Brower–Zadeh logic etc.) must be completed by additional laws (like the Solér law (Solér, 1995)) in order to achieve the Hilbert lattice and the classical Hilbert spaces. Furthermore, it must be clarified by means of the quantum theory of measurement (Busch *et al.*, 1996) whether for the unsharp propositions discussed here the nonobjectification problems disappear (Busch, 1998). Only if these questions were answered we could hope to obtain consistency between quantum language, quantum logic, and quantum mechanics.

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